The de Sitter scalar field equation has explicit dependence on the vector potential $A^\mu$, emerging from the minimal coupling condition. As a starting point for solving for $A^\mu$, it is known that the de Rham wave equation must be satisfied,

$$-A^{\alpha;\beta} + R^\alpha_{\beta\gamma} A^\gamma = \mu_0 J^\alpha$$

as well as the Lorentz gauge condition

$$A^\mu_{;\mu} = 0,$$

or, written in component form,

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}A^\alpha)_{,\alpha} = 0.$$

Features of the general FRW geometry can be utilized, where the metric is

$$ds^2 = -c^2 dt^2 + a^2(t)[d\chi^2 + \Sigma^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)$$

and where $a' = da/dt$,

$$G^t_t = -3\frac{a'^2 + kc^2}{a^2c^2}$$

$$G^{\chi}_\chi = G^\theta_\theta = G^\phi_\phi = -2\frac{a''}{a} - \frac{a'^2 + kc^2}{a^2c^2}.$$

Utilizing the relation $R^\alpha_\beta = G^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta G$, we see that

$$R^t_t = 3\frac{a''}{a}$$

$$R^{\chi}_\chi = R^{\theta}_\theta = R^{\phi}_\phi = \frac{a''}{a} + 2\frac{a'^2 + kc^2}{a^2c^2}.$$
From the Lagrangian
\[ \mathcal{L} = \frac{1}{2} \{ c^2 \dot{x}^2 - a^2(t) [ \dot{\chi}^2 + \Sigma^2(\chi) (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) ] \} , \]  
with \( \dot{x} = dx/d\tau \), the following Euler-Lagrange equations are derived,

\[
\begin{align*}
0 &= \ddot{t} + \frac{aa'}{c^2} [ \dot{\chi}^2 + \Sigma^2(\chi) (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) ] \\
0 &= \ddot{\chi} + 2 \frac{a'}{a} \dot{t} \dot{\chi} - \Sigma(\chi) \Sigma'(\chi) (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \\
0 &= \ddot{\theta} + 2 \frac{a'}{a} \dot{t} \dot{\theta} + 2 \frac{\Sigma'(\chi)}{\Sigma(\chi)} \dot{\chi} \dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 \\
0 &= \ddot{\phi} + 2 \frac{a'}{a} \dot{t} \dot{\phi} + 2 \frac{\Sigma'(\chi)}{\Sigma(\chi)} \dot{\chi} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} .
\end{align*}
\]

from which the connection coefficients \( \Gamma^\alpha_{\mu\gamma} \) can be read. For any FRW \( A^\mu \), the \( \Gamma^\alpha_{\mu\gamma} \)'s can be computed, along with the Ricci tensors \( R^\mu_{\nu\mu} \), to satisfy the conditions enforced by equations 1 and 2. Now, if a solution with Coulomb gauge (i.e. with \( A^t = 0 \)) is attempted and the spherical symmetry of de Sitter space is utilized to obtain \( A^\theta = A^\phi = 0 \), the simplicity of Gauss’s Law can be utilized,

\[ F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \]  
(9)
along with the gauge invariant Maxwell equation

\[ F^{\alpha\beta} ;_{\beta} = 0 \]  
(10)
to give the solution

\[ A^\chi = - \frac{\eta}{a^2(t) \Sigma^2(\chi)} . \]  
(11)
Notice that this solution satisfies equations 1 and 2. With the proper Coulomb normalization, the de Sitter vector potential solution is

\[ A^\chi = - \frac{Q \eta}{4\pi \epsilon_0 a^2(t) \Sigma^2(\chi)} . \]  
(12)
This solution is completely general for any FRW spacetime geometry. Attempting a solution with nonzero \( A^t \) leads to pure gauge, i.e. as solution that is the gradient of a solution to the d’Alembert equation. Also, the \( 1/\Sigma^2(\chi) \) dependence gives a singularity for \( \chi = \pi \), or at the
“south pole” in de Sitter space, implying the satisfaction of Gauss’s Law; that there is no such thing as an isolated charge in a closed universe.