First-Order Scalar Particle Scattering in de Sitter Space

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First-Order Scattering in QFT

de Sitter Space

QFT in de Sitter Space

New Calculations

Conclusions

(The notation will be used, $\hbar = c = 1$)
A free, relativistic scalar particle obeys the **Klein Gordon Equation**

\[
[\Box - m^2] \varphi(\vec{x}, t) = 0
\]

\[
\left(\Box = \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x^{\mu}}\right)
\]
Question: What does a wave scattering off of a potential look like in the remote future?

Answer: Propagators describe the time evolution of a wave...

\[ \varphi(\vec{x}', t') = i \int G(\vec{x}', t'; \vec{x}, t) \varphi(\vec{x}, t) \, d^3x \]

Under potential \( V(\vec{x}, t) \), the scattered wave is (to first-order)

\[ \psi(\vec{x}', t') = \varphi(\vec{x}', t') + i \int G(\vec{x}', t'; \vec{x}, t) V(\vec{x}, t) \varphi(\vec{x}, t) \, d^3x \, dt \]
Transition Amplitude

\[ |T_{fi}|^2 = \text{Probability to scatter from state } i \text{ to state } f \]

\[ T_{fi} = \delta_{fi} - i \int \varphi^*_f(\vec{x}', t') \, V(\vec{x}, t) \, \varphi_i(\vec{x}, t) \, d^3\!x \, dt \]
Feynman’s graphical representation of the first-order electromagnetic scattering interaction...

This interaction is of order-$\alpha$, where $\alpha$ is the fine structure constant ($\alpha \approx 1/137$). Higher order terms have been neglected for this calculation.
The Rutherford Formula

Relativistic Rutherford Scattering Formula

\[
\frac{d\sigma}{d\Omega} = \frac{q^2 Q^2}{(4\pi\epsilon_0)^2(4E)^2 \sin^4 \frac{\theta}{2}} \left[ 1 - \beta^2 \sin^2 \frac{\theta}{2} \right]
\]

\[
\beta = \frac{k}{\omega}
\]
de Sitter Space

Einstein Field Equations

\[ R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta} \]

de Sitter Space was a solution discovered in 1917 by Willem de Sitter (1872→1934) with \( \Lambda \neq 0 \) and \( T_{\alpha\beta} = 0 \)

- Empty, Closed Universe
- Positive \((k = +1)\) curvature
- Maximally Symmetric
- Obeys “Perfect Cosmological Principle”
de Sitter Space

\[ ds^2 = -c^2 dt^2 + a^2 \cosh^2 \left( \frac{t}{a} \right) \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right] \]

\[ d^4 V = a^3 \cosh^3 \left( \frac{t}{a} \right) \sin^2 \chi \sin \theta \, d\chi \, d\theta \, d\phi \, dt \]
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QFT in de Sitter Space

In de Sitter space...

\[ \Box \varphi = \frac{1}{\sqrt{-\det g}} \left( \sqrt{-\det g} g^{\alpha \beta} \varphi_{,\beta} \right)_{,\alpha} \]

Free standing waves exist as expansions in normal mode solutions of the de Sitter Klein Gordon equation, e.g.

\[ \varphi(t, \Omega_3) = a^{-3/2} \sum_L [b_L \chi_L(t) Y_L(\Omega_3) + H.c.] \]

- \( Y_L(\Omega_3) \) are 3-sphere harmonics
- \( \chi_L(t) \) describe time evolution of normal modes
$\mathcal{Y}_L(\Omega_3)$ and $\chi_L(t)$

$$\mathcal{Y}_L(\Omega_3) = \left[ (l_1+1) \frac{(l_1 + l_2 + 1)!}{(l_1 - l_2)!} \right]^{1/2} (\sin \chi)^{-1/2} P_{l_1+1/2}^{-l_2+1/2} (\cos \chi)$$

$$\times Y_{l_2}^m(\theta, \phi)$$

$$\chi_L(t) = \left[ \frac{a \pi/2}{\sinh \pi qa} \right]^{1/2} \cosh^{-3/2} \left( \frac{t}{a} \right)$$

$$\times \{ \kappa_L^{(+)} P_{l+1/2}^{-iqa} (\tanh (t/a)) + \kappa_L^{(-)} P_{l+1/2}^{+iqa} (\tanh (t/a)) \}$$
For minimal coupling to the electromagnetic potential, 
\[ p^\mu \to p^\mu - qA^\mu \] giving the modified energy-momentum relation

\[ (p_\mu - qA_\mu)(p^\mu - qA^\mu) + m^2 = 0 \]

If 2\textsuperscript{nd} order terms are neglected, the modified de Sitter scalar wave equation becomes

\[ \Box \phi - m^2 \phi = 2iqA^\mu \nabla_\mu \phi \]
The Electro “static” Potential

If Lorentz and Coulomb Gauge are imposed (i.e. $A_{\mu ; \mu} = 0$ & $A^0 = 0$), the vector potential takes the form $A^\mu = [0, A^\chi, 0, 0]$, where

$$A^\chi = -\frac{Q \eta}{4\pi \epsilon_0 a^2(t) \Sigma^2(\chi)}$$

and $\eta$ is the conformal time

$$\eta = \int^t \frac{dt'}{a(t')}$$
Calculation of $T_{fi}$ in de Sitter Space

The modified transition amplitude is then

$$T_{fi} = \delta_{fi} - i \int \varphi_f^*(t', \Omega_3') 2iq A^x \nabla x \varphi_i(t, \Omega_3) d^4V_{ds}$$

- A scattering selection rule for the angular-momentum quantum numbers has been derived...

$$l_1 \rightarrow l_1 + 2n - 1 \quad l_2 \rightarrow l_2 \quad m \rightarrow m \quad \Leftrightarrow \quad T_{fi} \neq 0$$

- A Simpson's rule numerical integration has been performed
Numerical Results, $l_1 = 1$
Numerical Results, $l_1 = 2$
Numerical Results, $l_1 = 3$
Conclusions

- Scattering strongly peaked at $l_1 \rightarrow l_1 \pm 1$
- No scattering from $l_1 = 0$ monopole state
- Unexpected probabilities for certain $l_1$ transitions
- In de Sitter space, solutions arise as scattering probabilities instead of the flat spacetime differential cross-section
References